

Perfect state transfer in XX chains induced by boundary magnetic fields

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A recent numerical study of short chains found near-perfect quantum state transfer between the boundary sites of a spin-1/2 XX chain if a sufficiently strong magnetic field acts on these sites. We show that the phenomenon is based on a pair of states strongly localized at the boundaries of the system and provide a simple quantitative analytical explanation.

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I. INTRODUCTION

The transfer of quantum states between different elements of a quantum computer is an important and challenging task in quantum information processing. Chains of interacting spin-1/2 particles were proposed as a means to achieve this task. Many different types of quantum spin chains have been discussed, as evident, for example, from the reviews 1 and 2. In the most frequently studied scenario the first site of a finite spin chain is prepared in a definite state which is then supposed to be transferred to the last site by the natural Hamiltonian dynamics.

Since the elementary excitations of spin chains are often dispersive, the initially sharply localized quantum state broadens as it propagates and thus quantum information degrades³ so that the state received is not identical to the state transmitted. In view of limits to fault tolerance in quantum information processing it is, however, desirable to achieve *perfect state transfer* (PST). Engineered chains were shown to be attractive means in reaching this goal^{2,4-9}. In these chains, all spin-spin couplings and / or local fields are set at certain values which guarantee PST.

However, from a practical point of view, it would be more desirable to achieve PST by simpler means, for example by manipulating only $\mathcal{O}(1)$ system parameters instead of the $\mathcal{O}(N)$ coupling constants which must be adjusted for an N -site engineered PST chain. This line of thought has been pursued in a number of studies in which the ends of a homogeneous spin chain, i.e., one with constant couplings, were connected to a single transmitting and receiving qubit, respectively, by adjustable coupling constants¹⁰⁻¹⁵ which influence the transfer of states via different mechanisms, depending on other details of the systems employed.

In a similar vein, Casaccino et al.¹⁶ suggested to use sufficiently strong magnetic fields acting exclusively on the boundary spins of an otherwise homogeneous spin chain. In their numerical results, they indeed observed near-perfect state transfer, however, with the transfer time growing strongly with both the chain length and the magnetic field strength.

In this paper, we explain the effect observed in Ref. 16 by analytical considerations. We show that boundary magnetic fields of sufficient strength h produce ex-

ceptional energy eigenstates of the chain. These states are exponentially localized at the boundaries with localization length $\sim (\ln h)^{-1}$. The PST is completely determined by these states and the dynamically relevant energy differences become exponentially small as the system size grows. Correspondingly the time scales become exponentially large, making this kind of PST unfortunately impractical for long chains. Furthermore, since the PST depends on an exponentially *weak link* between transmitter and receiver (the central region of the chain, where the amplitudes of the relevant states are exponentially small) one might speculate that any slight imperfection is bound to strongly affect the PST performance. The robustness of state transfer against disorder in the couplings was recently studied for several kinds of both engineered and boundary-controlled spin chains^{17,18}.

In Sec. II of this paper we present analytical solutions for the energy eigenstates and eigenvalues of the homogeneous semi-infinite chain with a magnetic field at the first site. The results thus obtained suggest a simple perturbative calculation and an approximate expression for the exact result for the finite chain with boundary fields at both ends, which we discuss in Sec. III and which explains the numerical observations in Ref. 16.

II. THE SEMI-INFINITE CHAIN: EXACT RESULTS

The spin-1/2 XX chain with boundary magnetic fields is defined by the Hamiltonian

$$H = \sum_{i=1}^{N-1} 2J (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + h (S_1^z + S_N^z), \quad (1)$$

where S_i^x , S_i^y , and S_i^z are the usual spin-1/2 operators with eigenvalues $\pm 1/2$. The most popular scenario in quantum information transport studies uses an initial product state, $|\psi(t=0)\rangle = |\varphi\rangle \otimes_{i=2}^N |\downarrow\rangle_i$, with $|\varphi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$, i.e., a superposition of the state with no spin up and a state with one single spin up located at site 1. Since the Hamiltonian conserves the number of spins ‘up’ and since the zero spin up component does not evolve in time it suffices to follow the evolution of the single spin up component. The corresponding N -dimensional sub-

space is spanned by the states $|i\rangle = S_i^+ |\downarrow \downarrow \dots \downarrow\rangle$ and the Hamiltonian matrix H_{ij} is given by

$$H_{i,i+1} = H_{i+1,i} = J; H_{11} = H_{NN} = h. \quad (2)$$

For $h = 0$ the eigenvectors \vec{u}^ν of the matrix (2) are given by their components

$$u_l^\nu = \left(\frac{2}{N+1} \right)^{\frac{1}{2}} \sin \frac{\nu\pi l}{N+1} \quad (3)$$

with $\nu = 1, \dots, N$, and the energy eigenvalues are

$$E_\nu = 2J \cos \frac{\nu\pi}{N+1}. \quad (4)$$

By the Hellmann-Feynman theorem^{19–22} all eigenvalues grow if a magnetic field $h > 0$ is switched on. To see more precisely what happens for $h \neq 0$ we now consider the case $N \rightarrow \infty$, i.e., we effectively study a semi-infinite chain with boundary magnetic field h . Since we are looking for states localized near the boundary $l = 1$ we use the Ansatz

$$u_l = e^{-\kappa l} \quad (5)$$

which turns out to be an eigenvector for $h > J$, with

$$\kappa = \ln \frac{h}{J} \quad (6)$$

and

$$E = 2J \cosh \kappa = h + \frac{J^2}{h}. \quad (7)$$

We thus find a critical field value, above which one energy eigenstate splits off from the quasi-continuum (4) and moves upward with growing h . The localization length of that state is proportional to $(\ln h/J)^{-1}$ and its amplitude at the boundary taking into account normalization is given by

$$(u_1)^2 = 1 - \left(\frac{J}{h} \right)^2. \quad (8)$$

The remaining eigenstates of the system are extended and similar to (3); the appropriate Ansatz is

$$u_l = \sin(ql + \delta), \quad (9)$$

where q is a wave vector between 0 and π . For finite N the value of q is fixed by the boundary condition at the far end of the system, leading to $q = \nu\pi/(N+1)$ for $h = 0$ as in Eq. (3), while for $N \rightarrow \infty$ q becomes continuous and can take any value between 0 and π . The state (9) yields the eigenvalue

$$E = 2J \cos q \quad (10)$$

and the phase shift δ obeys

$$\tan \delta = \frac{h \sin q}{J - h \cos q}. \quad (11)$$

Obviously δ has a singularity for $h > J$ at some q . However, that singularity is only a jump from $\pi/2$ to $-\pi/2$ (or vice versa) corresponding to an irrelevant sign change in the eigenvector components.

If N is finite (but large) the amplitudes of all extended eigenstates scale as $N^{-1/2}$, while the amplitude (8) of the localized state at the boundary site is constant and approaches unity as h grows. This is the key to the (quasi-) PST reported in Ref. 16. As N or h grow the localized states at the boundaries will increasingly dominate the transfer of localized quantum information between the ends of the chain, and we expect the speed of that transfer to be proportional to the overlap between the states localized at the left and right ends of the chain, respectively, which by (5) and (6) is proportional to $e^{-\kappa N} = \left(\frac{J}{h} \right)^N$.

III. THE FINITE CHAIN: APPROXIMATE RESULTS AND PERFECT STATE TRANSFER

We now consider the full model (1), with a strong boundary magnetic field $h \gg J$ at both ends of the N -site chain. We then have two extremely localized energy eigenstates with energies far above all other states. The system can thus be approximately described by two isolated spins in a strong field and an $(N-2)$ -site chain in between. The corresponding Hamiltonian matrix is obtained from (2) by setting

$$H_{12} = H_{21} = H_{N,N-1} = H_{N-1,N} = 0. \quad (12)$$

Taking parity into account, the two localized states are given by the amplitudes

$$u_1^\pm = \pm u_N^\pm = \frac{1}{\sqrt{2}}; u_l^\pm = 0 \text{ otherwise} \quad (13)$$

both states have energy $E_\pm = h$. The $N-2$ extended states are given by

$$u_1^\nu = u_N^\nu = 0; u_l^\nu = \left(\frac{2}{N-1} \right)^{\frac{1}{2}} \sin \frac{\nu\pi(l-1)}{N-1} \quad (14)$$

($\nu = 1, \dots, N-2$), with energy eigenvalues

$$E_\nu = 2J \cos \frac{\nu\pi}{N-1}. \quad (15)$$

Returning to the full problem defined by the Hamiltonian matrix (2) we expect modified energy eigenvalues E_\pm for the formerly degenerate isolated boundary states $|\pm\rangle$. The initial and final states of the desired perfect state transfer can be written in terms of the states (13):

$$|1\rangle = \frac{1}{\sqrt{2}} (|+\rangle + |-\rangle) \quad ; \quad |N\rangle = \frac{1}{\sqrt{2}} (|+\rangle - |-\rangle) \quad (16)$$

and consequently the transfer time τ is equal to the time during which a phase difference π develops between the

energy eigenstates $|+\rangle$ and $|-\rangle$, that is

$$\tau = \frac{\pi}{E_+ - E_-}. \quad (17)$$

The energy difference $E_+ - E_-$ can be determined approximately by treating the matrix elements J between sites 1 and 2 and sites $N-1$ and N , respectively, as perturbations. The corresponding perturbation operator V then is a $N \times N$ matrix with four nonzero entries J . The matrix elements of V between the unperturbed eigenstates (13) vanish and thus first-order degenerate perturbation theory does not lift the degeneracy. In second order, the corrections to the energy are given²³ by the eigenvalues of the matrix

$$\sum_{\nu \neq n, n'} \frac{\langle n|V|\nu\rangle\langle\nu|V|n'\rangle}{E_n^{(0)} - E_\nu^{(0)}}, \quad (18)$$

where $|n\rangle$ and $|n'\rangle$ are the unperturbed degenerate energy eigenstates with eigenvalue $E_n^{(0)} = h$, and $|\nu\rangle$ runs over all other unperturbed eigenstates (14), with eigenvalues $E_\nu^{(0)}$ (15). In our case the degenerate subspace is spanned by $|+\rangle$ and $|-\rangle$, or equivalently, by $|1\rangle$ and $|N\rangle$, compare (16). The matrix elements of the perturbation in the

basis $\{|1\rangle, |N\rangle\}$ are (compare (14))

$$\langle 1|V|\nu\rangle = Ju_2^\nu = J \left(\frac{2}{N-1} \right)^{\frac{1}{2}} \sin \frac{\nu\pi}{N-1} \quad (19)$$

and

$$\langle N|V|\nu\rangle = Ju_{N-1}^\nu = (-1)^{\nu+1} \langle 1|V|\nu\rangle. \quad (20)$$

The diagonal elements ($n = n'$) of the 2×2 matrix (18) thus turn out to be equal, leading to a mere energy shift. The non-diagonal elements are both equal to

$$\sum_{\nu=1}^{N-2} \frac{\langle 1|V|\nu\rangle\langle\nu|V|N\rangle}{h - 2J \cos \frac{\nu\pi}{N-1}} = \frac{2J^2}{N-1} \sum_{\nu=1}^{N-2} \frac{(-1)^{\nu+1} \sin^2 \frac{\nu\pi}{N-1}}{h - 2J \cos \frac{\nu\pi}{N-1}}, \quad (21)$$

leading to an energy splitting equal to twice the non-diagonal element (21):

$$E_+ - E_- = \frac{4J}{N-1} \sum_{\nu=1}^{N-2} \frac{(-1)^{\nu+1} \sin^2 \frac{\nu\pi}{N-1}}{\frac{h}{J} - 2 \cos \frac{\nu\pi}{N-1}}. \quad (22)$$

An asymptotic expansion of (22) in the small parameter J/h reads

$$E_+ - E_- = 2J \left(\frac{J}{h} \right)^{N-2} \left[1 + (N-3) \left(\frac{J}{h} \right)^2 + \frac{(N^2 - 3N - 2)}{2} \left(\frac{J}{h} \right)^4 + \mathcal{O} \left(\left(\frac{J}{h} \right)^6 \right) \right]. \quad (23)$$

This confirms the behavior which was to be expected based on the results of Sec. II and it explains the exponential growth of the transfer time (17) with both N and h/J .

The exact expression of the energy difference $E_+ - E_-$ can be found and approximated by the following simple consideration. The boundary-localized symmetric and antisymmetric eigenstates are determined using an ansatz of the form

$$u_l^\pm = e^{-\kappa l} \pm e^{-\kappa(N+1-l)}. \quad (24)$$

The corresponding energy eigenvalues are again given by

$$E_\pm = 2J \cosh \kappa_\pm, \quad (25)$$

(compare Eq.(7)) but now the values κ_\pm fulfill the transcendental equation

$$e^{\kappa_\pm} = \frac{h}{J} \pm e^{-N\kappa_\pm} \left(\frac{h}{J} e^{\kappa_\pm} - 1 \right) \quad (26)$$

which results from the condition for the eigenvalue at sites 1 or N . This equation can be solved iteratively,

starting from $\kappa_0 = \ln \frac{h}{J}$ (compare Eq.(6)). To lowest order in $(J/h)^{N-2}$ this yields

$$E_+ - E_- = 2J \left(\frac{J}{h} \right)^{N-2} \left(1 - \left(\frac{J}{h} \right)^2 \right) + \mathcal{O} \left(\left(\frac{J}{h} \right)^{2N-4} \right). \quad (27)$$

For comparison to the numerical results of Ref. 16 we identify h with Δ_E from that reference and set $J = 2$.

h	Ref. 16	exact diag.	perturbative	order $(J/h)^{N-2}$
10	99	107	90	107
20	8010	801	770	801
30	2665	2674	2627	2674
40	6260	6315	6252	6315
50	12294	12311	12233	12311

TABLE I: Transfer times for a chain of length $N = 5$ with different values of the boundary field h as obtained by various methods. See text for details.

In Table I we compare the transfer time for $N = 5$ as obtained by different methods and for different values of h . The column labelled “Ref. 16” contains the numbers reported in that reference (with an obvious typing error for $h = 20$). The next column shows the transfer time

(17) from the difference of the two dominant eigenvalues E_+ and E_- as obtained from an exact numerical diagonalization of the Hamiltonian (2), whereas the fourth column is obtained from the perturbation calculation described above.

The asymptotic expansion (23) agrees to the expression (22) to a much higher precision than that of the numbers in the table. However, it can be seen that the perturbative result approaches the exact one only for fairly large values of h . On the other hand, the lowest order in $(\frac{J}{h})^{N-2}$, given in Eq. (27), agrees with the exact values to all digits given (compare the last column in Table I).

We have performed similar comparisons for chain lengths $N = 7$ and $N = 12$, obtaining similar results, that is, for sufficiently strong boundary fields the perturbation calculation yields the transfer time to a precision of one percent or better while the lowest order fits the exact result within 10^{-7} for $N = 7$ and within 10^{-14} for $N = 12$, as was to be expected from Eq. (27). However, at $N = 12$ the difference between the two leading energy eigenvalues approaches the precision limits of standard numerical procedures and the corresponding transfer times become prohibitively long by any standards.

Finally, we briefly discuss the degree of fidelity which can be reached by the scheme discussed here. A reasonable measure of the transfer fidelity between the states $|1\rangle$ and $|N\rangle$ is

$$f_{1N}(t) = |\langle 1|e^{-iHt}|N\rangle|^2. \quad (28)$$

Employing the energy eigenstates $|\nu\rangle$ and the eigenvalues E_ν we can write

$$\begin{aligned} \langle 1|e^{-iHt}|N\rangle &= \sum_{\nu} \langle 1|\nu\rangle \langle \nu|N\rangle e^{-iE_\nu t} \\ &= \langle 1|+\rangle \langle +|N\rangle e^{-iE_+ t} \\ &\quad + \langle 1|-\rangle \langle -|N\rangle e^{-iE_- t} \\ &\quad + \sum_{\mu} \langle 1|\mu\rangle \langle \mu|N\rangle e^{-iE_\mu t}, \end{aligned} \quad (29)$$

where we have separated the dominant (localized) eigenstates $|\pm\rangle$ from the remaining (extended) eigenstates $|\mu\rangle$, each of which yields a contribution $\mathcal{O}(1/N)$, with a quasi-random phase. The transfer time τ is the instant when the two dominant terms in (29) interfere constructively, maximizing f_{1N} and thus leading to (compare (8))

$$|\langle 1|e^{-iH\tau}|N\rangle| \approx 1 - \left(\frac{J}{h}\right)^2 \quad (30)$$

or

$$f_{1N}(\tau) = 1 - 2 \left(\frac{J}{h}\right)^2. \quad (31)$$

The dominant time dependence of $f_{1N}(t)$ is a slow oscillation between zero and the maximum value (31) with period 2τ , with superimposed rapid small-amplitude oscillations from the subdominant terms in (29). This is precisely what numerical calculations for small N show.

The value $f_{1N}(\tau)$ does not depend on the chain length N and it grows (increasingly slowly) as h grows. Note, however, that the transfer time τ grows exponentially as h grows. For a given N the choice of h thus is the result of a trade-off between transfer time (17) and fidelity (31), depending on the relative importance assigned to each of these two figures of merit. Hence, our analytical results for transfer time and fidelity provide the basis for finding an optimum trade-off in practice.

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